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# FIRST INTEGRAL AND PHASE PORTRAIT FOR A CLASS OF TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS

**Rachid Boukoucha**[1]

[1]University of Bejaia, Bejaia, Algeria

**Abstract.** In this paper we are interested in studying the existence of the first integral and to the curves which are formed by the trajectories of the 2-dimensional differential systems. Concrete example exhibiting the applicability of our result is introduced.

***Keywords:*** *Autonomous system, first integral, periodic orbits, limit cycle.*

***AMS Subject Classification:*** *34C05, 34C07, 37C27, 37K10.*

***Corresponding Author:*** *Rachid Boukoucha, Department of Technology, Faculty of Technology, University of Bejaia, 06000, Bejaia, Algeria, e-mail: rachid_boukecha@yahoo.fr*

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## 1. Introduction

We consider two-dimensional autonomous systems of differential equations of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = F(x,y), \\ y' = \dfrac{dy}{dt} = G(x,y), \end{cases} \tag{1}$$

where $F(x,y)$ and $G(x,y)$ are real's functions. In the qualitative theory of planar dynamical systems [1, 8, 9], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem [18]. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly [2, 3, 6, 11, 13]. There exist three main open problems in the qualitative theory of real planar differential systems, the distinction between a centre and a focus, the determination of the number of limit cycles and their distribution, and the determination of its integrability. The importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals. One of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals. Giné and Llibre characterized a large classes of polynomial differential systems in terms of the existence of first integrals [4, 7, 9, 12, 16, 17, 21]. For more details about first integral see for instance [10, 14, 15, 19, 20] see the references quoted in those articles. We recall that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1).

System (1) is integrable on an open set $\Omega$ of $\mathbb{R}^2$ if there exists a non constant $C^1$ function $H:\Omega\to\mathbb{R}$, called first integral of the system on $\Omega$, which is constant on the trajectories of the system (1) contained in $\Omega$, i.e. if

$$\frac{dH(x,y)}{dt}=\frac{\partial H(x,y)}{\partial x}F(x,y)+\frac{\partial H(x,y)}{\partial y}G(x,y)\equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H=h$ is the general solution of this equation, where $h$ is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of the first integral determines their phase portrait (see [5]).

In this paper we are interested in studying the existence of the first integral and to the curves which are formed by the trajectories of the 2-dimensional differential systems of the form

$$\begin{cases} x'=B_1(x,y)+xB_3(x,y)\cos\left(\frac{A_1(x,y)}{A_2(x,y)}\right), \\ y'=B_2(x,y)+yB_3(x,y)\cos\left(\frac{A_1(x,y)}{A_2(x,y)}\right), \end{cases} \tag{2}$$

where $A_1(x,y)$, $A_2(x,y)$, $B_1(x,y)$, $B_2(x,y)$, $B_3(x,y)$ are homogeneous polynomials of degree $a$, $a$, $n$, $n$, $m$ respectively.

We define the trigonometric functions

$$f_1(\theta)=B_1(\cos\theta,\sin\theta)(\cos\theta)+B_2(\cos\theta,\sin\theta)(\sin\theta),$$

$$f_2(\theta)=B_3(\cos\theta,\sin\theta)\cos\left(\frac{A_1(\cos\theta,\sin\theta)}{A_2(\cos\theta,\sin\theta)}\right),$$

$$f_3(\theta)=(\cos\theta)B_2(\cos\theta,\sin\theta)-(\sin\theta)B_1(\cos\theta,\sin\theta).$$

## 2. Main result

Our main result on the existence of the first integral and the curves which are formed by the trajectories of the 2-dimensional differential systems (2) is the following.

**Theorem 1.** Consider a two-dimensional differential system (2), then the following statements hold.

(1) If $f_3(\theta)\neq 0$, $A_2(\cos\theta,\sin\theta)\neq 0$ and $n-m\neq 2$, then system (2) has the first integral

$$H(x,y)=\left(x^2+y^2\right)^{\frac{n-m-1}{2}}\exp\left((m-n+1)\int^{\arctan\frac{y}{x}}M(\omega)\,d\omega\right)-$$

$$(n-m-1)\int^{\arctan\frac{y}{x}}\exp\left((m-n+1)\int^{w}M(\omega)\,d\omega\right)N(w)\,dw,$$

where $M(\theta)=\frac{f_1(\theta)}{f_3(\theta)}$, $N(\theta)=\frac{f_2(\theta)}{f_3(\theta)}$, and the phase portrait of the differential system (2), in Cartesian coordinates is given by

$$x^2+y^2=\begin{pmatrix} h\exp\left((n-m-1)\int^{\arctan\frac{y}{x}}M(\omega)d\omega\right)+ \\ (n-m-1)\exp\left((n-m-1)\int^{\arctan\frac{y}{x}}M(\omega)d\omega\right) \\ \int^{\arctan\frac{y}{x}}\exp\left((m-n+1)\int^{w}M(\omega)d\omega\right)N(w)dw \end{pmatrix}^{\frac{2}{n-m-1}},$$

where $h\in\mathbb{R}$. Moreover, the system (2) has no periodic orbit in one of open quadrants.

(2) If $f_3(\theta)\neq 0$, $A_2(\cos\theta,\sin\theta)\neq 0$ and $n-m=2$, then system (2) has the first integral

$$H(x,y)=\left(x^2+y^2\right)^{\frac{1}{2}}\exp\left(-\int^{\arctan\frac{y}{x}}\left(M(\omega)+N(\omega)\right)d\omega\right),$$

and the phase portrait of the differential system (2) , in Cartesian coordinates is given by

$$\left(x^2+y^2\right)^{\frac{1}{2}}-h\exp\left(\int^{\arctan\frac{y}{x}}\left(M(\omega)+N(\omega)\right)d\omega\right)=0,$$

where $h\in\mathbb{R}$. Moreover, the system (2) has no limit cycle.

(3) If $f_3(\theta)=0$ for all $\theta\in\mathbb{R}$, then system (2) has the first integral $H=\frac{y}{x}$, and the phase portrait of the differential system (2), in Cartesian coordinates is given by $y-hx=0$, where $h\in\mathbb{R}$. Moreover, the system (2) has no limit cycle.

**Proof.** In order to prove our results we write two-differential system (2) in polar coordinates $(r,\theta)$, defined by $x=r\cos\theta$ and $y=r\sin\theta$, then system (2) becomes

$$\begin{cases} r'=f_1(\theta)r^n+f_2(\theta)r^{m+1}, \\ \theta'=f_3(\theta)r^{n-1}, \end{cases} \tag{3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction, $r'=\frac{dr}{dt}$ and $\theta=\frac{d\theta}{dt}$. If $f_3(\theta)\neq 0$, $A_2(\cos\theta,\sin\theta)\neq 0$ and $n-m\neq 2$.

Taking as independent variable the coordinate $\theta$ , differential system (3) is written as

$$\frac{dr}{d\theta}=M(\theta)r+N(\theta)r^{2+m-n}, \tag{4}$$

where $M(\theta)=\frac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta)=\frac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variables $\rho=r^{n-m-1}$ we obtain the linear equation

$$\frac{d\rho}{d\theta}=(n-m-1)\left(M(\theta)\rho+N(\theta)\right). \tag{5}$$

The general solution of linear equation (5) is

$$\rho(\theta)=\exp\left((n-m-1)\int^{\theta}M(\omega)d\omega\right)$$
$$\left(\mu+(n-m-1)\int^{\theta}\exp\left((n-m-1)\int^{w}M(\omega)d\omega\right)N(w)dw\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{n-m-1}{2}} \exp\left( (m-n+1)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega \right) +$$

$$(m-n+1)\int^{\arctan\frac{y}{x}} \exp\left( (m-n+1)\int^{w} M(\omega)\,d\omega \right) N(w)\,dw.$$

Let $\Gamma$ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$x^2+y^2 = \left( \begin{array}{c} h\exp\left( (n-m-1)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega \right) + \\ (n-m-1)\exp\left( (n-m-1)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega \right) \\ \int^{\arctan\frac{y}{x}} \exp\left( (m-n+1)\int^{w} M(\omega)\,d\omega \right) N(w)\,dw \end{array} \right)^{\frac{2}{n-m-1}},$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit $\Gamma$ is contained in the curve

$$x^2+y^2 = \left( \begin{array}{c} h_\Gamma\exp\left( (n-m-1)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega \right) + \\ (n-m-1)\exp\left( (n-m-1)\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega \right) \\ \int^{\arctan\frac{y}{x}} \exp\left( (m-n+1)\int^{w} M(\omega)\,d\omega \right) N(w)\,dw \end{array} \right)^{\frac{2}{n-m-1}}.$$

But this curve cannot contain the periodic orbit $\Gamma$ and consequently no limit cycle contained in the one of open quadrants, because this curve in each open quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in \mathbb{R}$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in \mathbb{R}$, the abscissa is given by

$$x = \frac{1}{\sqrt{1+\eta^2}} \left( \begin{array}{c} h_\Gamma\exp\left( (n-m-1)\int^{\arctan\eta} M(\omega)\,d\omega \right) + \\ (n-m-1)\exp\left( (n-m-1)\int^{\arctan\eta} M(\omega)\,d\omega \right) \\ \int^{\arctan\eta} \exp\left( (m-n+1)\int^{w} M(\omega)\,d\omega \right) N(w)\,dw \end{array} \right)^{\frac{2}{n-m-1}},$$

at most a unique value of $x$, consequently at most a unique point in each pen quadrant. So this curve cannot contain the periodic orbit.

Hence statement (1) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$, $A_2(\cos\theta, \sin\theta) \neq 0$ and $n - m = 2.$

Taking as independent variable the coordinate $\theta$, differential system (3) is written as

$$\frac{dr}{d\theta} = M(\theta)r + N(\theta), \tag{6}$$

where $M(\theta)=\frac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta)=\frac{f_2(\theta)}{f_3(\theta)}$, which is a linear equation.

The general solution of linear equation (6) is

$$\rho(\theta)=\exp\left(\int^{\theta}M(\omega)d\omega\right)$$
$$\left(\mu+\int^{\theta}\exp\left(-\int^{w}M(\omega)d\omega\right)N(w)dw\right),$$

where $\mu\in\mathrm{R}$, which has the first integral

$$H(x,y)=\sqrt{x^2+y^2}\exp\left(-\int^{\arctan\frac{y}{x}}M(\omega)d\omega\right)-$$
$$-\int^{\arctan\frac{y}{x}}\exp\left(-\int^{w}M(\omega)d\omega\right)N(w)dw.$$

Let $\Gamma$ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma=H(\Gamma)$.

The curves $H=h$ with $h\in\mathrm{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$x^2+y^2=\left(\begin{matrix}h\exp\left(\int^{\arctan\frac{y}{x}}M(\omega)d\omega\right)+\exp\left(\int^{\arctan\frac{y}{x}}M(\omega)d\omega\right)\\ \int^{\arctan\frac{y}{x}}\exp\left(-\int^{w}M(\omega)d\omega\right)N(w)dw\end{matrix}\right)^2,$$

where $h\in\mathrm{R}$.

Therefore the periodic orbit $\Gamma$ is contained in the curve

$$x^2+y^2=\left(\begin{matrix}h_\Gamma\exp\left(\int^{\arctan\frac{y}{x}}M(\omega)d\omega\right)+\exp\left(\int^{\arctan\frac{y}{x}}M(\omega)d\omega\right)\\ \int^{\arctan\frac{y}{x}}\exp\left(-\int^{w}M(\omega)d\omega\right)N(w)dw\end{matrix}\right)^2.$$

But this curve cannot contain the periodic orbit $\Gamma$ and consequently no limit cycle contained in the one of open quadrants, because this curve in each open quadrant has at most a unique point on every straight line $y=\eta x$ for all $\eta\in\mathrm{R}$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y=\eta x$ for all $\eta\in\mathrm{R}$, the abscissa is given by

$$x=\frac{1}{\sqrt{1+\eta^2}}\left(\begin{matrix}h_\Gamma\exp\left(\int^{\arctan\eta}M(\omega)d\omega\right)+\exp\left(\int^{\arctan\eta}M(\omega)d\omega\right)\\ \int^{\arctan\eta}\exp\left(-\int^{w}M(\omega)d\omega\right)N(w)dw\end{matrix}\right)^2,$$

at most a unique value of $x$, consequently at most a unique point in each open quadrant. So this curve cannot contain the periodic orbit.

Hence statement (2) of Theorem 1 is proved.

Assume now that $f_3(\theta)=0$ for all $\theta\in\mathrm{R}$. Then from system (3) it follows that $\theta'=0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, consequently no limit cycle.

This completes the proof of statement (3) of Theorem 1.

**Example 1.**

The following example is given to illustrate our result.

If we take $A_1(x,y) = 5x^2 + 4y^2$, $A_2(x,y) = x^2 + y^2$,
$B_1(x,y) = x^4 + x^3y + 2x^2y^2 + xy^3 + y^4$, $B_2(x,y) = x^4y + 2x^3y^2 + 2x^2y^3 + 2xy^4 + y^5$ and $B_3(x,y) = 3x^2 - xy + 3y^2$, then system (2) turns to

$$\begin{cases} x' = \left(x^5 + x^4y + 2x^3y^2 + x^2y^3 + xy^4\right) + x\left(3x^2 - xy + 3y^2\right)\cos\left(\frac{5x^2+4y^2}{x^2+y^2}\right), \\ y' = \left(x^4y + 2x^3y^2 + 2x^2y^3 + 2xy^4 + y^5\right) + y\left(3x^2 - xy + 3y^2\right)\cos\left(\frac{5x^2+4y^2}{x^2+y^2}\right), \end{cases}$$

The two-dimensional differential system in polar coordinates $(r,\theta)$ becomes

$$r' = \left(1 + \tfrac{3}{4}\sin 2\theta - \tfrac{1}{8}\sin 4\theta\right)r^5 + \left(3 - \cos\theta\sin\theta\right)\cos\left(\tfrac{9}{2} + \tfrac{1}{2}\cos 2\theta\right)r^3,$$
$$\theta' = \left(\cos^2\theta\sin^2\theta\right)r^4,$$

here $f_1(\theta) = 1 + \frac{3}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta$, $f_2(\theta) = (3 - \cos\theta\sin\theta)\cos\left(\frac{9}{2} + \frac{1}{2}\cos 2\theta\right)$ and $f_3(\theta) = \cos^2\theta\sin^2\theta$, it is the case (1) of the Theorem 1, then this two-dimensional differential system has the first integral

$$H(x,y) = \left(x^2 + y^2\right)\exp\left(-2\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) - 2\int^{\arctan\frac{y}{x}} \exp\left(-2\int^{w} M(\omega)\,d\omega\right) B(w)\,dw,$$

where $M(\omega) = \dfrac{1 + \frac{3}{4}\sin 2\omega - \frac{1}{8}\sin 4\omega}{\cos^2\omega\sin^2\omega}$, $N(w) = \dfrac{(3 - \cos w\sin w)\cos\left(\frac{9}{2} + \frac{1}{2}\cos 2w\right)}{\cos^2 w\sin^2 w}$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of this differential system, in Cartesian coordinates are written as

$$x^2 + y^2 = h\exp\left(2\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + 2\exp\left(2\int^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\int^{\arctan\frac{y}{x}} \exp\left(-2\int^{w} N(\omega)\,d\omega\right) N(w)\,dw,$$

where $h \in \mathbb{R}$. This system has no periodic orbits, and consequently no limit cycle**.**

## 3. Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar rational differential systems of ODEs in order to obtain explicit expression for a first integral and characterizes its trajectories; this is a one of the classical tools in the classification of all trajectories of dynamical systems.

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